A Coupled-Mode Approach to the Analysis of Fields in Space-Curved and Twisted Waveguides

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Abstract —**A coupled-mode approach for solving Maxwell's equations in terms of unitary and reciprocal unitary vectors is deduced in a twisted space-curved coordinate system. Application of this method to the analysis of fields in a space-curved and twisted single-mode optical fiber and a twisted rectangular microwave waveguide is presented and compared with the results from existing literature.**

I. Introduction

RECENTLY, the development of optical fiber technology and the application of infrared lasers have led to growing interest in an in-depth analysis of the wave propagation along space-curved and twisted waveguides. Lewin *et al.* [1] have given a general summary of the available analysis in this field. Because of the lack of appropriate mathematical tools, most of the analyses are limited to the pure bending case (with zero torsion) [2]–[5] using the toroidal coordinate system and the pure twisted case [6]–[9] using physical intuition. However, space-curved and twisted waveguides are encountered more often in practical applications.

The basic feature of a space-curved and twisted waveguide is that the centerline of the waveguide is a space curve which can be described by a position vector $\boldsymbol{R}(s)$, a function of s, the arc length measured from an arbitrary point [1]. The cross section of the waveguide at any value of s is maintained in the same form as that of the original straight waveguide. The orientation of the cross section could coincide with the unit normal vector $\boldsymbol{a}_n$ and the unit binormal vector $\boldsymbol{a}_b$ of the Serret–Frenet frame or could deviate from them by a certain angle. Only in the latter case is the waveguide regarded as being twisted.

In the space configuration mentioned above, different coordinate systems could be used to solve Maxwell's equations. Sollfrey [10] first used a nonorthogonal curvilinear coordinate system. Pierce *et al.* [11] used the "sheath" method. These analyses are mainly concerned with the field in the vicinity of a helical wire. The present authors [12] have analyzed the field in helically wound optical fibers by using the local orthogonal curvilinear coordinate system introduced by Tang [13], [1]. This method does not seem to be very effective in solving field problems for waveguides with noncircular cross sections. Lewin [14], [1] derived the scalar wave equation to study twisted rectangular waveguides by using the helical coordinate system. The nonorthogonal system composed of the unitary and reciprocal unitary vectors has been used by Yabe *et al.* [15] in solving the fields of dominant hybrid modes in a twisted rectangular waveguide. However, the perturbational feature of that method makes it ineffective in those cases where the power of one mode will fully transfer into another mode.

In this paper, an attempt has been made to use a coupled-mode approach to solve the guided-wave problems in space-curved and twisted waveguides by using the unitary and reciprocal unitary vector coordinate system. The field equations and the formulation of the coupled-mode equations will be described in the subsequent sections, and the applications of this method to the fields in a space-curved single-mode optical fiber and a twisted rectangular microwave waveguide will be presented as an initial test and verification of this method. Problems which need further investigation will be discussed.

II. The Unitary and Reciprocal Vectors in a Twisted Space-Curved Coordinate System

As shown in Fig. 1, the centerline of the waveguide is a smooth space curve, which can be described by a position vector $\boldsymbol{R}(s)$, a function of s, the arc length measured from an arbitrary point. The cross section of the waveguide on a plane perpendicular to the space curve at any point with its arc length s should be precisely the same as that of the original straight waveguide. The orientation of the cross section could deviate from the Serret–Frenet vectors $\boldsymbol{a}_n$ and $\boldsymbol{a}_b$ [1] by an angle Ⓗ, with $Ⓗ = \int_0^s \alpha\, ds$. Here α, a function of s, is defined as the twist rate of the waveguide. In this paper, we introduce the twisted coordinates (X, Y, S), in which axes X and Y are in the plane of $\boldsymbol{a}_n$ and $\boldsymbol{a}_b$ of the space curve but rotate through an angle Ⓗ relative to the Serret–Frenet vectors $\boldsymbol{a}_n$ and $\boldsymbol{a}_b$ with S equal to s, i.e.,

$$\begin{bmatrix} X \\ Y \\ S \end{bmatrix} = \begin{bmatrix} \cos Ⓗ & \sin Ⓗ & 0 \\ -\sin Ⓗ & \cos Ⓗ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n \\ b \\ s \end{bmatrix} \qquad (1)$$

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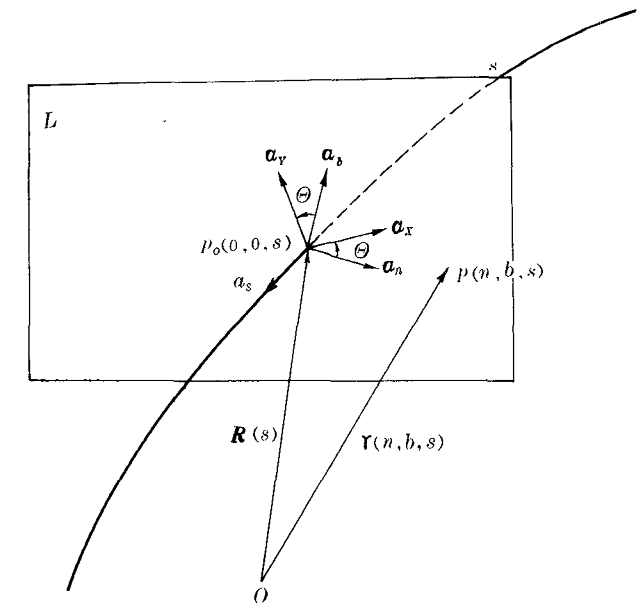

Fig. 1. The Serret–Frenet frame $(\boldsymbol{a}_n, \boldsymbol{a}_b, \boldsymbol{a}_s)$ and the twisted space-curved coordinate system (X, Y, S). Plane L is perpendicular to the space curve. The twist angle is Θ, with $\Theta = \int_0^s \alpha\, ds$, where $\alpha(s)$ is the twist rate of the waveguide.

where n, b, and s are the coordinates in the Serret–Frenet frame [1]. The unitary and reciprocal unitary vectors can be derived according to the formal procedure [16]. For the position vector of an arbitrary point $\boldsymbol{r} = \boldsymbol{R}(s) + n\boldsymbol{a}_n + b\boldsymbol{a}_b$ [1], the unitary vectors will be

$$\begin{aligned} \boldsymbol{a}_1 &= \partial \boldsymbol{r}/\partial X = \boldsymbol{a}_n \cos\Theta + \boldsymbol{a}_b \sin\Theta = \boldsymbol{a}_X \\ \boldsymbol{a}_2 &= \partial \boldsymbol{r}/\partial Y = -\boldsymbol{a}_n \sin\Theta + \boldsymbol{a}_b \cos\Theta = \boldsymbol{a}_Y \\ \boldsymbol{a}_3 &= \partial \boldsymbol{r}/\partial S = -b(\alpha+\tau)\boldsymbol{a}_n + n(\alpha+\tau)\boldsymbol{a}_b + (1-\chi n)\boldsymbol{a}_s \\ &= -Y(\alpha+\tau)\boldsymbol{a}_X + X(\alpha+\tau)\boldsymbol{a}_Y \\ &\quad + \left[1 - \chi\left(X\cos\Theta - Y\sin\Theta\right)\right]\boldsymbol{a}_s \end{aligned} \tag{2}$$

where χ is the curvature and τ is the torsion of the centerline of the waveguide [1].

Let $V = \boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)$. We have $V = 1 - \chi n$. The reciprocal unitary vectors will be

$$\begin{aligned} \boldsymbol{a}^1 &= (\boldsymbol{a}_2 \times \boldsymbol{a}_3)/V = \boldsymbol{a}_X + \frac{Y(\alpha+\tau)}{V}\boldsymbol{a}_s \\ \boldsymbol{a}^2 &= (\boldsymbol{a}_3 \times \boldsymbol{a}_1)/V = \boldsymbol{a}_Y - \frac{X(\alpha+\tau)}{V}\boldsymbol{a}_s \\ \boldsymbol{a}^3 &= (\boldsymbol{a}_1 \times \boldsymbol{a}_2)/V = \boldsymbol{a}_s/V. \end{aligned} \tag{3}$$

The relevant metric coefficients can be derived according to the relation $g_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j$:

$$\begin{aligned} &g_{11} = 1 \qquad g_{22} = 1 \qquad g_{33} = V^2 + (\alpha+\tau)^2(X^2+Y^2) \\ &g_{12} = g_{21} = 0 \qquad g_{13} = g_{31} = -(\alpha+\tau)Y \\ &g_{23} = g_{32} = (\alpha+\tau)X. \end{aligned} \tag{4}$$

It is important to mention that the normal to the waveguide wall at a certain point is always contained within the plane defined by the vectors $\boldsymbol{a}^1$ and $\boldsymbol{a}^2$ at that point. The reason for that is simple. In order for the cross section of the waveguide at any arc length s on the centerline to remain invariant in the X, Y coordinate system, the equation of the surface of the waveguide wall should be independent of S; i.e., it can be written in a form $\phi(X,Y) = 0$, where $\phi(X,Y)$ is a definite function of the two variables. The normal to the waveguide wall will be [16]

$$\nabla\phi(X,Y) = \boldsymbol{a}^1\, \partial\phi/\partial X + \boldsymbol{a}^2\, \partial\phi/\partial Y. \tag{5}$$

That means the normal to the waveguide wall is in the plane defined by $\boldsymbol{a}^1$ and $\boldsymbol{a}^2$.

Accordingly, for a straight waveguide with the same cross section, the equation of the guide wall will be $\phi(x,y) = 0$, with x, y the Cartesian coordinates, and the normal to the guide wall will be

$$\nabla\phi(x,y) = \boldsymbol{i}_x\, \partial\phi/\partial x + \boldsymbol{i}_y\, \partial\phi/\partial y \tag{6}$$

where $\boldsymbol{i}_x$ and $\boldsymbol{i}_y$ are the unit vectors along the x and y axes in the Cartesian coordinate system.

III. Maxwell's Equations and the Coupled-Mode Formulation

We resolve the electric field into its covariant components and the magnetic field into its contravariant components such that

$$\boldsymbol{E} = \sum_{i=1}^{3} E_i \boldsymbol{a}^i \qquad \boldsymbol{H} = \sum_{i=1}^{3} H^i \boldsymbol{a}_i. \tag{7}$$

Using the expression for the curl of a vector with respect to a set of general coordinates [16] and the relations between the covariant and contravariant components and the relevant vectors

$$H_i = \sum_{j=1}^{3} g_{ij} H^j \qquad \boldsymbol{a}_i = \sum_{j=1}^{3} g_{ij} \boldsymbol{a}^j \tag{8}$$

we can derive Maxwell's equations in the unitary and reciprocal unitary coordinate system and obtain

$$\left(\frac{\partial E_3}{\partial Y} - \frac{\partial E_2}{\partial S}\right)\boldsymbol{a}_1 + \left(\frac{\partial E_1}{\partial S} - \frac{\partial E_3}{\partial X}\right)\boldsymbol{a}_2 + \left(\frac{\partial E_2}{\partial X} - \frac{\partial E_1}{\partial Y}\right)\boldsymbol{a}_3 = -j\omega\mu_0 V\left(H^1\boldsymbol{a}_1 + H^2\boldsymbol{a}_2 + H^3\boldsymbol{a}_3\right) \tag{9}$$

$$\begin{aligned} &\left[\frac{\partial(H^3V^2)}{\partial Y} - \frac{\partial H^2}{\partial S} + I^1\right]\boldsymbol{a}^1 + \left[\frac{\partial H^1}{\partial S} - \frac{\partial(H^3V^2)}{\partial X} + I^2\right]\boldsymbol{a}^2 \\ &+ \left[V^2\left(\frac{\partial H^2}{\partial X} - \frac{\partial H^1}{\partial Y}\right) + I^3\right]\boldsymbol{a}^3 \\ &= j\omega\epsilon V\left(E_1\boldsymbol{a}^1 + E_2\boldsymbol{a}^2 + E_3\boldsymbol{a}^3\right) \end{aligned} \tag{10}$$

with

$$I^1=(\alpha+\tau)\left(X\frac{\partial H^2}{\partial Y}-Y\frac{\partial H^2}{\partial X}-H^1-X\frac{\partial H^3}{\partial S}\right)+(\alpha+\tau)^2\left(X^2\frac{\partial H^3}{\partial Y}-XY\frac{\partial H^3}{\partial X}\right) \tag{11}$$

$$I^2=(\alpha+\tau)\left(Y\frac{\partial H^1}{\partial X}-X\frac{\partial H^1}{\partial Y}-H^2-Y\frac{\partial H^3}{\partial S}\right)+(\alpha+\tau)^2\left(-Y^2\frac{\partial H^3}{\partial X}+XY\frac{\partial H^3}{\partial Y}\right) \tag{12}$$

$$I^3=(\alpha+\tau)\left(X\frac{\partial H^1}{\partial S}+Y\frac{\partial H^2}{\partial S}+2VH^3\right)+(\alpha+\tau)^2\left[-X^2\frac{\partial H^1}{\partial Y}+Y^2\frac{\partial H^2}{\partial X}+XY\left(\frac{\partial H^1}{\partial X}-\frac{\partial H^2}{\partial Y}\right)+YH^1-XH^2\right]. \tag{13}$$

In the subsequent part of this section, the formal coupled-mode analysis procedure in [17] is used to treat the above Maxwell's equations in the waveguide. The total covariant components E_1 and E_2 and the total contravariant components H^1 and H^2 of the transverse fields are expanded into series of the transverse fields of the ideal modes in a straight waveguide with the same cross section as that of the space-curved and twisted waveguide.

The ideal modes are characterized as

$$E_i(\nu)=e_{i(\nu)}(X,Y)\exp(-\gamma_\nu S)$$
$$H^i_{(\nu)}=h^i_{(\nu)}(X,Y)\exp(-\gamma_\nu S)\qquad (i=1,2,3) \tag{14}$$

where ν is used to label the modes.

The total transverse field components of the actual waveguide are expressed as the sum of the field components of the ideal modes with proper coefficients A_ν and B_ν, which are functions of the arc length s only. It is understood that the propagation factor $\exp(-\gamma_\nu S)$ is included in these coefficients. We have

$$E_i=\sum_\nu A_\nu(S)e_{i(\nu)}(X,Y)$$
$$H^i=\sum_\nu B_\nu(S)h^i_{(\nu)}(X,Y)\qquad (i=1,2). \tag{15}$$

The field functions of the ideal modes satisfy the following Maxwell's equations:

$$\left(\frac{\partial e_{3(\nu)}}{\partial Y}+\gamma_\nu e_{2(\nu)}\right)\boldsymbol{a}_1+\left(-\gamma_\nu e_{1(\nu)}-\frac{\partial e_{3(\nu)}}{\partial X}\right)\boldsymbol{a}_2=-j\omega\mu_0\left(h^1_{(\nu)}\boldsymbol{a}_1+h^2_{(\nu)}\boldsymbol{a}_2\right) \tag{16}$$

$$\left(\frac{\partial e_{2(\nu)}}{\partial X}-\frac{\partial e_{1(\nu)}}{\partial Y}\right)\boldsymbol{a}_3=-j\omega\mu_0 h^3_{(\nu)}\boldsymbol{a}_3 \tag{17}$$

$$\left(\frac{\partial h^3_{(\nu)}}{\partial Y}+\gamma_\nu h^2_{(\nu)}\right)\boldsymbol{a}^1+\left(-\gamma_\nu h^1_{(\nu)}-\frac{\partial h^3_{(\nu)}}{\partial X}\right)\boldsymbol{a}^2=j\omega\epsilon\left(e_{1(\nu)}\boldsymbol{a}^1+e_{2(\nu)}\boldsymbol{a}^2\right) \tag{18}$$

$$\left(\frac{\partial h^2_{(\nu)}}{\partial X}-\frac{\partial h^1_{(\nu)}}{\partial Y}\right)\boldsymbol{a}^3=j\omega\epsilon e_{3(\nu)}\boldsymbol{a}^3. \tag{19}$$

Here, we emphasize again that only the total transverse field components of the waveguide are expanded into series of ideal modes, while the longitudinal components E_3 and H^3 should be derived from Maxwell's equations (9) and (10) such that

$$E_3=\left[V\left(\frac{\partial H^2}{\partial X}-\frac{\partial H^1}{\partial Y}\right)+I^3/V\right]\Big/j\omega\epsilon=\sum_\nu\left[VB_\nu e_{3(\nu)}+J_\nu\right] \tag{20}$$

$$H^3=\left(\frac{\partial E_2}{\partial X}-\frac{\partial E_1}{\partial Y}\right)\Big/(-j\omega\mu_0 V)=\sum_\nu A_\nu h^3_{(\nu)}/V \tag{21}$$

where

$$J_\nu=(\alpha+\tau)\left[\left(Xh^1_{(\nu)}+Yh^2_{(\nu)}\right)\frac{dB_\nu}{dS}+2A_\nu h^3_{(\nu)}\right]\Big/(j\omega\epsilon V)+O\left[(\alpha+\tau)^2X\right]. \tag{22}$$

We now substitute the field expansions (15), (20), and (21) into Maxwell's equations (9) and (10) and using mode equations (16)–(19), they may be expressed as

$$\sum_\nu\left\{\left[-\left(\frac{dA_\nu}{dS}+B_\nu\gamma_\nu\right)e_{2(\nu)}+\chi\gamma_\nu B_\nu\left(X\cos Ⓗ-Y\sin Ⓗ\right)e_{2(\nu)}+\chi B_\nu\sin Ⓗ e_{3(\nu)}+\frac{\partial J_\nu}{\partial Y}\right]\boldsymbol{a}_1\right.$$
$$\left.+\left[\left(\frac{dA}{dS}+B_\nu\gamma_\nu\right)e_{1(\nu)}-\chi\gamma_\nu B_\nu\left(X\cos Ⓗ-Y\sin Ⓗ\right)e_{1(\nu)}+\chi B_\nu\cos Ⓗ e_{3(\nu)}-\frac{\partial J_\nu}{\partial X}\right]\boldsymbol{a}_2\right\}=0 \tag{23}$$

$$\sum_\nu\left\{\left[-\left(\frac{dB_\nu}{dS}+A_\nu\gamma_\nu\right)h^2_{(\nu)}+\chi\gamma_\nu A_\nu\left(X\cos Ⓗ-Y\sin Ⓗ\right)h^2_{(\nu)}+\chi A_\nu\sin Ⓗ h^3_{(\nu)}+I^1_\nu\right]\boldsymbol{a}^1\right.$$
$$\left.+\left[\left(\frac{dB_\nu}{dS}+A_\nu Y_\nu\right)h^1_{(\nu)}-\chi\gamma_\nu A_\nu\left(X\cos Ⓗ-Y\sin Ⓗ\right)h^1_{(\nu)}+\chi A_\nu\cos Ⓗ h^3_{(\nu)}+I^2_\nu\right]\boldsymbol{a}^2\right\}=0 \tag{24}$$

where

$$I_\nu^1 = (\alpha+\tau)\left[B_\nu\left(X\frac{\partial h^2_{(\nu)}}{\partial Y} - Y\frac{\partial h^2_{(\nu)}}{\partial X} - h^1_{(\nu)}\right) - \frac{X}{V}h^3_{(\nu)}\frac{dA_\nu}{dS}\right] + 0\left[(\alpha+\tau)^2 X\right] \quad (25)$$

$$I_\nu^2 = (\alpha+\tau)\left[B_\nu\left(Y\frac{\partial h^1_{(\nu)}}{\partial X} - X\frac{\partial h^1_{(\nu)}}{\partial Y} - h^2_{(\nu)}\right) - \frac{Y}{V}h^3_{(\nu)}\frac{dA_\nu}{dS}\right] + 0\left[(\alpha+\tau)^2 X\right]. \quad (26)$$

In order to proceed further with the derivation of a coupled-mode equation system, we need the orthogonality relation of the ideal modes

$$\int_{A_\infty}\left(e_{1(\nu)}h^2_{(\mu)} - e_{2(\nu)}h^1_{(\mu)}\right)dA = P_\nu\delta_{\mu\nu} \quad (27)$$

where A_∞ indicates the infinite cross section and $\delta_{\mu\nu}$ the Kronecker delta for discrete indices μ and ν.

We take the scalar product of (23) with $(h^1_{(\mu)}\boldsymbol{a}^1 + h^2_{(\mu)}\boldsymbol{a}^2)$ and (24) with $(e_{1(\mu)}\boldsymbol{a}_1 + e_{2(\mu)}\boldsymbol{a}_2)$, and use the relation $\boldsymbol{a}_i\cdot\boldsymbol{a}^j = \delta_{ij}$. After integration over the infinite cross section, we obtain, with the help of the orthogonality relation (27),

$$\frac{dA_\mu}{dS} + \gamma_\mu B_\mu = \sum_\nu\int_{A_\infty}\left[-\frac{\partial J_\nu}{\partial Y}h^1_{(\mu)} + \frac{\partial J_\nu}{\partial X}h^2_{(\mu)} + \chi G_{\nu\mu}B_\nu\right]dA/P_\mu \quad (28)$$

$$\frac{dB_\mu}{dS} + \gamma_\mu A_\mu = \sum_\nu\int_{A_\infty}\left[I^1_\nu e_{1(\mu)} + I^2_\nu e_{2(\mu)} + \chi F_{\nu\mu}A_\nu\right]dA/P_\mu \quad (29)$$

where

$$G_{\nu\mu} = \gamma_\nu\left(X\cos Ⓗ - Y\sin Ⓗ\right)\left(-e_{2(\nu)}h^1_{(\mu)} + e_{1(\nu)}h^2_{(\mu)}\right) - \left(\sin Ⓗ\, h^1_{(\mu)} + \cos Ⓗ\, h^2_{(\mu)}\right)e_{3(\nu)}$$

$$F_{\nu\mu} = \gamma_\nu\left(X\cos Ⓗ - Y\sin Ⓗ\right)\left(h^2_{(\nu)}e_{1(\mu)} - h^1_{(\nu)}e_{2(\mu)}\right) + \left(\sin Ⓗ\, e_{1(\mu)} + \cos Ⓗ\, e_{2(\mu)}\right)h^3_{(\nu)}. \quad (30)$$

As the transverse fields are composed of waves traveling in the positive s and negative s directions, we introduce the transformation

$$A_\mu = C^+_\nu + C^-_\mu \qquad B_\mu = C^+_\mu - C^-_\mu \quad (31)$$

where C^+_μ denotes the complex amplitude of the forward-traveling wave and C^-_μ that of the backward-traveling wave.

Substituting (31) into (28) and (29), adding and subtracting these two equations, we finally obtain the following coupled-mode equations

$$\frac{dC^+_\mu}{dS} + \gamma_\mu C^+_\mu = \sum_\nu\int_{A_\infty}\left[-\frac{\partial J_\nu}{\partial Y}h^1_{(\mu)} + \frac{\partial J_\nu}{\partial X}h^2_{(\mu)} + I^1_\nu e_{1(\mu)} + I^2_\nu e_{2(\mu)} + \chi\left(A_\nu F_{\nu\mu} + B_\nu G_{\nu\mu}\right)\right]dA/(2P_\mu) \quad (32)$$

$$\frac{dC^-_\mu}{dS} - \gamma_\mu C^-_\mu = \sum_\nu\int_{A_\infty}\left[-\frac{\partial J_\nu}{\partial Y}h^1_{(\mu)} + \frac{\partial J_\nu}{\partial X}h^2_{(\mu)} - I^1_\nu e_{1(\mu)} - I^2_\nu e_{2(\mu)} + \chi\left(-A_\nu F_{\nu\mu} + B_\nu G_{\nu\mu}\right)\right]dA/(2P_\mu). \quad (33)$$

It is worthwhile to note that the expansion of the transverse covariant and contravariant components of the fields in a space-curved and twisted waveguide into series of the transverse fields of the ideal modes in a straight waveguide is rational in the following boundary condition consideration. As described in the last part of Section II, we can see that if a vector $A\boldsymbol{i}_x + B\boldsymbol{i}_y$ is perpendicular to the wall of a straight waveguide at a certain point on the boundary of the cross section, then $A\boldsymbol{a}^1 + B\boldsymbol{a}^2$ will be perpendicular to the wall of the curved and twisted waveguide with the same cross section. Using the relations of multiplication of unitary and reciprocal unitary vectors, we can easily prove that the transverse fields E_1, E_2, H^1, H^2 will automatically satisfy the boundary conditions when they are expanded into series of corresponding field components of the ideal modes in the straight waveguide.

IV. Application to Wave Propagation Problems in Single-Mode Optical Fibers

We consider a simple case where we assume that there are only two polarization modes HE^X_{11} and HE^Y_{11} with $\mathrm{HE}^{\pm XT}_{11}$, etc., denoting the transverse fields of the ideal modes. The total transverse field in the curved and twisted optical fiber is

$$C^+_X\left(\mathrm{HE}^{+XT}_{11}\right) + C^-_X\left(\mathrm{HE}^{-XT}_{11}\right) + C^+_Y\left(\mathrm{HE}^{+YT}_{11}\right) + C^-_Y\left(\mathrm{HE}^{-YT}_{11}\right). \quad (34)$$

Substituting the standard field expression [18] in a step index single-mode fiber into (32) and (33), after a laborious but straightforward derivation, we obtain

$$\frac{dC^+_Y}{dS} + j\beta_Y C^+_Y = (\alpha+\tau)\left[-C^+_X + C^-_X/2 + 0\left(1/\beta^2_Y a^2\right)\right] \quad (35)$$

$$\frac{dC^+_X}{dS} + j\beta_X C^+_X = (\alpha+\tau)\left[C^+_Y - C^-_Y/2 + 0\left(1/\beta^2_X a^2\right)\right] \quad (36)$$

$$\frac{dC^-_Y}{dS} - j\beta_Y C^-_Y = (\alpha+\tau)\left[-C^-_X + C^+_X/2 + 0\left(1/\beta^2_Y a^2\right)\right] \quad (37)$$

$$\frac{dC^-_X}{dS} - j\beta_X C^-_X = (\alpha+\tau)\left[C^-_Y - C^+_Y/2 + 0\left(1/\beta^2_X a^2\right)\right] \quad (38)$$

where β_X and β_Y denote the propagation constants of the HE_{11}^X and HE_{11}^Y modes. At $s=0$, we assume $C_X^-|_{s=0} \doteq C_Y^-|_{s=0} \doteq 0$. Then because $(\beta_X+\beta_Y) \gg (\alpha+\tau)$ and $\beta^2 a^2 \gg 1$, the solution of the above equations will be $C_X^- \doteq 0$, $C_Y^- \doteq 0$, and

$$\frac{dC_X^+}{dS} \doteq -j\beta_X C_X^+ + (\alpha+\tau)C_Y^+ \tag{39}$$

$$\frac{dC_Y^+}{dS} \doteq -j\beta_Y C_Y^+ - (\alpha+\tau)C_X^+. \tag{40}$$

These equations have been verified by experiments and are generally acknowledged [6]–[9], [19], [20], so the derivation of (39) and (40) from (35)–(38) can be regarded as a test of the validity of the theory in the case of optical fibers.

V. Application to Twisted Microwave Rectangular Waveguides

A procedure similar to that in the last section may be applied in the case of twisted microwave rectangular waveguides in which the total transverse fields are expanded into series of ideal modes TE_{mn} and TM_{mn} with $\mathrm{TE}_{mn}^{\pm\mathrm{T}}$ etc. denoting the transverse fields of the ideal modes. The total transverse field is

$$\sum_{mn}\left[C_{mn}^+(\mathrm{TE}_{mn}^{+\mathrm{T}}) + C_{mn}^-(\mathrm{TE}_{mn}^{-\mathrm{T}}) + C_{(mn)}^+(\mathrm{TM}_{mn}^{+\mathrm{T}}) + C_{(mn)}^-(\mathrm{TM}_{mn}^{-\mathrm{T}})\right]. \tag{41}$$

Substituting the standard field expressions of the different modes [21] in a rectangular waveguide with infinite conductive walls into (32) and (33), we find that the modes which may be excited by the dominant TE_{10} mode are TE_{pq} and TM_{pq}, with p even and q odd, and TE_{0n}, with n odd. This is in good agreement with [15].

Calculations of the coupling coefficients in (32) and (33) show that the main higher order mode which could be excited by the dominant TE_{10} mode is the TE_{01} mode. Other modes are relatively small. In order to obtain a clear physical picture, we neglect for the moment all higher order modes other than the TE_{01} mode, and assume that the TE_{01} mode is cut off in the straight waveguide with the same cross section. After straightforward derivations according to (32) and (33), we obtain

$$\frac{dC_{01}^+}{dS} + \gamma_{01}C_{01}^+ = \tau(M_1+M_2)/(2P_{01}) \tag{42}$$

$$\frac{dC_{01}^-}{dS} - \gamma_{01}C_{01}^- = \tau(M_1-M_2)/(2P_{01}) \tag{43}$$

$$\frac{dC_{10}^+}{dS} + \gamma_{10}C_{10}^+ = \tau(N_1+N_2)/(2P_{10}) \tag{44}$$

$$\frac{dC_{10}^-}{dS} - \gamma_{10}C_{10}^- = \tau(N_1-N_2)/(2P_{10}) \tag{45}$$

where γ_{01} is a positive real value $\gamma_{01} = |(-k^2+\pi^2/b^2)^{1/2}|$, while $\gamma_{10} = j\beta_{10} = j|(k^2-\pi^2/a^2)^{1/2}|$ is an imaginary value; a and b denote the cross-sectional dimensions and $k=2\pi/\lambda$, with λ the space wavelength of the wave. M_1, M_2, N_1, N_2 are coefficients determined by the field amplitude C_{10}^+, C_{10}^- and other constants, i.e.,

$$M_1 = \frac{-8\pi^2\gamma_{01}}{j\omega\epsilon\mu_0^2 a^2}\left[(C_{10}^+ + C_{10}^-) + 0(\tau a)\right] \tag{46}$$

$$M_2 = \frac{-8j\omega\gamma_{10}}{\mu_0}\left[(C_{10}^+ - C_{10}^-) + 0(\tau a)\right] \tag{47}$$

$$N_1 = \frac{8\pi^2\gamma_{10}}{j\omega\epsilon\mu_0^2 b^2}\left[(C_{01}^+ + C_{01}^-) + 0(\tau a)\right] \tag{48}$$

$$N_2 = \frac{8j\omega\gamma_{01}}{\mu_0}\left[(C_{01}^+ - C_{01}^-) + 0(\tau a)\right] \tag{49}$$

while

$$P_{01} = j\pi^2\omega a\gamma_{01}/(2\mu_0 b) \tag{50}$$

$$P_{10} = j\pi^2\omega b\gamma_{10}/(2\mu_0 a). \tag{51}$$

At $s=0$, we assume $C_{10}^-|_{s=0} \doteq 0$; then the solution of (42) and (43) will be

$$C_{01}^+ + C_{01}^- \doteq \frac{\tau(j\beta_T M_1 + \gamma_{01}M_2)}{P_{01}(\gamma_{01}^2+\beta_T^2)}C_{10}^+ \tag{52}$$

$$C_{01}^+ - C_{01}^- \doteq \frac{\tau(j\beta_T M_2 + \gamma_{01}M_1)}{P_{01}(\gamma_{01}^2+\beta_T^2)}C_{10}^+ \tag{53}$$

$$C_{10}^- < (\tau a/2\pi)^2 C_{10}^+ \tag{54}$$

$$j\beta_T \doteq j\beta_{10} + 16\tau a(C_{01}^+ + C_{01}^-)/(k^2 b^3). \tag{55}$$

In determining the value of $(C_{01}^+ + C_{01}^-)$, etc., we could use the approximation $\beta_T \doteq \beta_{10}$.

Taking a rectangular waveguide as an example with $a=2.29$ cm, $b=1.02$ cm, $\lambda=3$ cm, and a twist rate $\alpha=\pi/11.4$ rad/cm and substituting these values into (52), we find that the relative amplitude of the electric fields of the TE_{01} mode to the dominant TE_{10} mode aA_{01}/bA_{10} is

$$\frac{a(C_{01}^+ + C_{01}^-)}{b(C_{10}^+ + C_{10}^-)} \doteq -j0.05 \qquad (C_{10}^- < 0.01C_{10}^+) \tag{56}$$

which is generally consistent with the result in [15]. However, the value of $C_{01}^+ - C_{01}^-$, the relative amplitude of the magnetic field of the TE_{01} mode, is different.

In the above example, we have neglected all higher modes other than TE_{01}. In fact, no difficulty will occur if more higher order modes have to be taken into account. The propagation constant β_T [22] of the hybrid mode can also be obtained. In short, we can derive the hybrid-mode fields and also the modification of its propagation constants from the coupled-mode equations in twisted rectangular waveguides. When the cross section of the waveguide becomes more and more square such that the TE_{01} mode propagates [23], then γ_{01} will be imaginary. In this case the coupled-mode approach still works, but the solutions will have features similar to that of optical fibers, and the TE_{10} and TE_{01} modes will couple. The capability of this method for solving fields problems in nearly square guides has the same advantage as the coupled-mode approach.

VI. Discussion

The coupled-mode approach to field problems in space-curved and twisted waveguides appears to be attractive because of its ease of physical interpretation and flexibility in application. Two examples have been given in this paper as an initial test of the validity of this method. However, the mathematical rigor needs to be further investigated. For example only the transverse components of the fields are matched in the coupled-mode approach. When the coupled-mode equations are used to analyze the field distortion in curved and twisted open waveguides such as an optical fiber, the radiation continuum modes or leaky modes should be introduced into the calculation. The methodology of how to handle the continuum mode requires investigation.

Coupled-mode theory has been widely used in field problems of straight waveguides with perturbation of their cross section or refractive indices [17]. The introduction of the new method to solving problems associated with space-curved and twisted waveguides appears to be promising.

References

[1] J. Lewin, D. C. Chang, and E. F. Keuster, *Electromagnetic Wave and Curved Structures*. Southgate House, England: Peter Pergrinus Ltd., 1977.

[2] D. Marcuse, "Field deformation and loss caused by curvature of optical fibers," *J. Opt. Soc. Am.*, vol. 44, pp. 311–320, Apr. 1976.

[3] M. E. Marhic, L. I. Kwan, and M. Epstein, "Optical surface waves along a toroidal metallic guide," *Appl. Phys. Lett.*, vol. 33, pp. 609–611, Oct. 1978.

[4] X-S. Fang and Z-Q. Lin, "Birefringence in curved single-mode optical fibers due to waveguide geometry effect—Perturbation analysis," *J. Lightwave Technol.*, vol. LT-3, pp. 789–793, Aug. 1985.

[5] J. Sakai and T. Kimura, "Analytical bending loss formula of optical fibers with field deformation," *Radio Sci.*, vol. 17, pp. 21–29, Jan. 1982.

[6] P. McIntyre and A. W. Snyder, "Light propagation in twisted aninstropic media: Application to photorecepter," *J. Opt. Soc. Am.*, vol. 68, pp. 149–157, Feb. 1978.

[7] T. Okoshi, K. Kikuchi, and K. Emura, "Rotation of polarization in single-mode optical fibers," IECE Tech. Rep., OQE80-62, July 1980 (in Japanese).

[8] M. Monerie and L. Jeunhomme, "Polarization mode coupling in long single-mode fibres," *Opt. Quantum Electron.*, vol. 12, pp. 449–461, Nov. 1980.

[9] R. Ulrich and A. Simon, "Polarization optics of twisted single-mode fibers," *Appl. Opt.*, vol. 18, pp. 2241–2251, July 1979.

[10] W. Sollfrey, "Wave propagation on helical wires," *J. Appl. Phys.*, vol. 22, pp. 905–910, July 1951.

[11] J. R. Piece and P. K. Tien, "Coupling of modes in helixes," *Proc. IRE*, vol. 42, pp. 1389–1396, Sept. 1954.

[12] X-S. Fang and Z-Q. Lin, "Field in single-mode helically-wound optical fibers," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-33, pp. 1150–1154, Nov. 1985.

[13] C. H. Tang, "An orthogonal coordinate system for curved pipes," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, p. 69, Jan. 1970.

[14] L. Lewin, "Propagation in curved and twisted waveguides of rectangular cross-section," *Proc. Inst. Elec. Eng.*, vol. 102, pt. B, pp. 75–80, Jan. 1955.

[15] H. Yabe and Y. Mushiake, "An analysis of a hybrid-mode in a twisted rectangular waveguide," *IEEE Trans. Microwave Theory Tech.* vol. MTT-32, pp. 65–71, Jan. 1984.

[16] J. A. Stratton, *Electromagnetic Theory*. New York: McGraw-Hill, 1941.

[17] D. Marcuse, *Theory of Dielectric Optical Waveguides*. New York: Academic Press, 1974.

[18] T. Okoshi, *Optical Fibers*. New York: Academic Press, 1982.

[19] J. N. Ross, "The rotation of the polarization in low birefringence monomade optical fibres due to geometric effects," Opt. Quantum Electron., *vol.* 16, *pp.* 455–461, *Nov.* 1984.

[20] A. Papp and H. Harms, "Polarization optics of liquid-core optical fibers," *Appl. Opt.*, vol. 16, pp. 1315–1319, Aug. 1977.

[21] M. J. Adams, *An Introduction to Optical Waveguides*. New York: Wiley, 1981.

[22] H. Yabe, K. Nishio, and Y. Mushiake "Dispersion characteristics of twisted rectangular waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp. 91–96, Jan. 1984.

[23] L. Lewin and T. Ruehle, "Propagation in twisted square waveguide," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 44–48, Jan. 1980.

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